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First steps towards exact algebraic identification

Christiane Hespel[a,*], Gérard Jacob[b]

[a] I.N.S.A. Laboratoire L.A.N.S., 20, avenue des Buttes de Coësmes, 35043 Rennes Cédex, France
[b] L.I.F.L.(U.A. 369 C.N.R.S.), Bat.M3, Université Lille I, 59655 Villeneuve d'Ascq Cédex, France

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Abstract

This paper presents the first step towards solving the problem of the exact algebraic identification of causal functionals. This problem consists in computing the coefficients of a noncommutative generating series when only the Taylor expansion of some inputs (at $t=0$), and the Taylor expansion (at $t=0$) of associated outputs are known.

Resumé

Cet article présente la première étape vers la résolution du problème de l'Identification Algébrique Exacte des fonctionnelles causales. Ce problème consiste à calculer les coefficients d'une série génératrice non commutative, connaissant les développements de Taylor (en $t=0$) des entrées, et les développements de Taylor (en $t=0$) des sorties correspondantes.

1. Introduction

The causal functionals as defined by Fliess [3] are obtained from noncommutative generating series by evaluating words as iterated integrals of the inputs. In this paper, we are interested in solving the 'algebraic identification problem'.

> Is there an algorithm for computing the coefficients of a generating series when the Taylor expansions (at $t=0$) of the system inputs and of related outputs are known?

A positive answer would introduce the causal functionals as fully combinatorial objects. Some proofs are actually known of the injectivity of the 'evaluation map' [11,4,9]: if two generating series G and H define the same causal functional, then actually $G=H$. But they do not solve our problem. Here, first we recall and briefly discuss them. Then, we present a computation formula for the iterated derivatives of the output.

* Corresponding author. E-mail: hespel@nimbus.irisa.fr.

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Finally, we present a first step of the algebraic identification: we provide an algorithm computing the contribution, in the generating series G, of each inputs multiderivative. It will remain (in a future work) to compute the coefficient in G of each inputs iterated integral.

2. Chen series

2.1. Preliminaries and notations

See [3,5,7,10]. We associate to the input $a=(a_0,\dots,a_m)$ (with $a_0(t)\equiv 1$) an encoding alphabet $\mathscr{Z}=\{z_0,z_1,\dots,z_m\}$. A *word* w in $\mathscr{Z}^*$ is a finite sequence of letters in $\mathscr{Z}$. We denote by ε the *empty word*. A *noncommutative formal series* S on $\mathscr{Z}$ is a mapping of $\mathscr{Z}^*$ into $\mathbb{R}$

$$S=\sum_{w\in Z^*}\langle S\mid w\rangle w.$$

The operations of sum and Cauchy product of two series S, T are defined by

$$S+T=\sum_{w\in Z^*}(\langle S\mid w\rangle+\langle T\mid w\rangle)w; \qquad S.T=\sum_{w\in Z^*}\sum_{uv=w}\langle S\mid u\rangle.\langle T\mid v\rangle w.$$

We define as follows the *iterated integral* $\langle\mathscr{C}_a(t)\mid w\rangle=\int_0^t\delta_a(w)$ of a word w for input a:

$$\int_0^t\delta_a(\varepsilon)=1,$$

$$\int_0^t\delta_a(vz_i)=\int_0^t\left(\int_0^\tau\delta_a(v)\right)a_i(\tau)\,\mathrm{d}\tau,\quad \forall z_j\in Z,\ \forall v\in Z^*.$$

Hence, we define *the Chen series* [1] of the input a as being $\mathscr{C}_a=\sum_{w\in\mathscr{Z}^*}\langle\mathscr{C}_a\mid w\rangle w$. The time derivative of $\mathscr{C}_a$ (see [10]) is given as $(\mathrm{d}/\mathrm{d}t)\mathscr{C}_a=\mathscr{C}_a.\mathscr{L}_a$ where we have set $\mathscr{L}_a=\sum_{0\leqslant i\leqslant m}a_i.z_i$. Otherwise, *any noncommuting power series* H over $\mathscr{Z}$, the coefficients of which satisfy the following '*convergence condition*': $\exists K,L\in\mathbb{R}^+|\langle H\mid w\rangle|<K|w|!L^{|w|}$, $\forall w\in Z^*$ defines a causal functional [3], absolutely convergent in a neighbourhood of 0, with the output

$$y_H(t)=\sum_{w\in\mathscr{Z}^*}\langle H\mid w\rangle\langle\mathscr{C}_a(t)\mid w\rangle=\langle H\,\|\,\mathscr{C}_a(t)\rangle,$$

where $\|$ means infinite sum. From the derivation formula of Chen series, we deduce

$$\dot{y}_H(t)=\langle H\,\|\,\mathscr{C}_a.\mathscr{L}_a\rangle.$$

The concatenation of an input a defined on $[0,t_1[$ and b defined on $[0,t_2[$ is the input $a\sharp b$ defined on $[0,t_1+t_2[$ by

$$a\sharp b(t)=\begin{cases}a(t) & \text{if } 0\leqslant t<t_1,\\ b(t-t_1) & \text{if } t_1\leqslant t<t_1+t_2.\end{cases}$$

Then we have the relation $\mathscr{C}_{a\sharp v}(t_1+t)=\mathscr{C}_a(t_1)\mathscr{C}_v(t)$. We have also

$$\begin{array}{ll} \text{for } 0<t<t_1, & \dot{y}(t)=\langle G \parallel \mathscr{C}_a(t).\mathscr{L}_a(t)\rangle, \\ \text{for } t_1\leqslant t_1+t<t_1+t_2, & \dot{y}(t)=\langle G \parallel \mathscr{C}_a(t_1).\mathscr{C}_b(t).\mathscr{L}_b(t)\rangle, \\ \text{and then} & \lim_{t\to 0^+}[\dot{y}(t_1+t)]=\langle G \parallel \mathscr{C}_a(t_1).\mathscr{L}_b(0^+)\rangle. \end{array}$$

2.2. Sketch of three proofs about injectivity

Proposition 2.1. *The generating series evaluation of the input/output functionals is* faithful. *In other words, any series indistinguishable from* 0 *is identically null.*

We cursorily recall three proofs of this proposition, in relation to the identification problem.

2.2.1. The proof of Sontag and Wang [11]

This proof is based on noncommutative parameters partial derivatives. We set

$$b=\mu_1\sharp\mu_2\sharp\cdots\sharp\mu_k,$$

the input $\mu_i=(\mu_{i,1},\mu_{i,2},\ldots,\mu_{i,m})$ is defined on $[0,t_i[$.

The Chen series of the input b is $\mathscr{C}_b=\mathscr{C}_{\mu_1}\mathscr{C}_{\mu_2}\cdots\mathscr{C}_{\mu_k}$. We consider the output $y(t)$ as a function of the t_j. Then, its iterated partial noncommuting derivatives must be computed as follows:

$$\frac{\partial}{\partial t_k}(\langle G \parallel \mathscr{C}_{\mu_1}\cdots\mathscr{C}_{\mu_k}\rangle)_{t_k=0^+}=\langle G \parallel \mathscr{C}_{\mu_1}\cdots\mathscr{C}_{\mu_{k-1}}\mathscr{L}_{\mu_k}(0^+)\rangle,$$

$$\frac{\partial}{\partial t_{k-1}}\frac{\partial}{\partial t_k}(\langle G \parallel \mathscr{C}_{\mu_1}\cdots\mathscr{C}_{\mu_k}\rangle)_{t_k=0^+,t_{k-1}=0^+}=\langle G \parallel \mathscr{C}_{\mu_1}\cdots\mathscr{C}_{\mu_{k-2}}\mathscr{L}_{\mu_{k-1}}(0^+)\mathscr{L}_{\mu_k}(0^+)\rangle.$$

By iterating and assuming G indistinguishable from 0, we obtain

$$\langle G \parallel \mathscr{L}_{\mu_1}(0^+)\cdots\mathscr{L}_{\mu_k}(0^+)\rangle=\sum_{l_1,\ldots,l_k}\langle G \mid z_{l_1}\cdots z_{l_k}\rangle\mu_{1,l_1}\mu_{2,l_2}\cdots\mu_{k,l_k}=0$$

for any values of μ_{i,l_i}. Then $G=0$, since its coefficients can be recovered as follows:

$$\langle G \parallel z_{j_1}\cdots z_{j_k}\rangle=\frac{\partial^k}{\partial_{\mu_{1,j_1}}\cdots\partial_{\mu_{k,j_k}}}\left(\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}(y)\right)_{t_1=0^+}=0.$$

2.2.2. The proof of Fliess [4]

For $\varepsilon>0$, ε small, there are inputs a,b and a_i $(i\neq 0)$, the Chen series of which are

$$\mathscr{C}_a=\mathrm{e}^{\varepsilon z_0};\qquad \mathscr{C}_b=\mathrm{e}^{2\varepsilon z_0};\qquad \mathscr{C}_{a_i}=\mathrm{e}^{\varepsilon z_0+\varepsilon^2 z_i}.$$

Then we verify,

$$z_0=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\mathscr{C}_b-\mathscr{C}_a];\qquad z_i=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}[\mathscr{C}_{a_i}-\mathscr{C}_a].$$

Now, G being indistinguishable from zero, we deduce for any Chen series $\mathscr{C}$

$$\langle z_0 \triangleleft G \,\|\, \mathscr{C} \rangle = \langle G \,\|\, \mathscr{C} z_0 \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\langle G \,\|\, \mathscr{C}.\mathscr{C}_b \rangle - \langle G \,\|\, \mathscr{C}\mathscr{C}_a \rangle] = 0,$$

$$\forall i \in [1,m], \quad \langle z_i \triangleleft G \,\|\, \mathscr{C} \rangle = \langle G \,\|\, \mathscr{C} z_i \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} [\langle G \,\|\, \mathscr{C}.\mathscr{C}_{a_i} \rangle - \langle G \,\|\, \mathscr{C}\mathscr{C}_a \rangle] = 0.$$

Then, the power series $z_0 \triangleleft G$ and $z_i \triangleleft G$ are indistinguishable from zero. Recursively, for any word w the series $w \triangleleft \mathscr{C}$ also is indistinguishable from zero. It follows that the series G is equal to zero, because we have $\forall w \in Z^* \langle G \mid w \rangle = \langle w \triangleleft G \mid \varepsilon \rangle = 0$.

2.2.3. The proof of Reutenauer [9]

Shortly, one computes first a concatenated input of the form

$$\mathscr{C} = \mathscr{C}_{a_1}(t_1) \cdots \mathscr{C}_{a_p}(t_p) = \mathrm{e}^{t_1 z_0 + \alpha_1 z_{i_1}} \cdots \mathrm{e}^{t_p z_0 + \alpha_p z_{i_p}}.$$

By noting that $y(t) = \langle G \,\|\, \mathscr{C} \rangle \equiv 0$, by expanding $\mathscr{C}$ as power series in the parameters t_i and α_i, and by using properties of analytical functions of real variables, one finds that

$$\langle G \mid z_0^{j_0} z_1 \cdots z_0^{j_{p-1}} z_{i_{p-1}} z_0^{j_p} \rangle = 0.$$

2.2.4. Remarks on these proofs

All these proofs require the knowledge of $y(t)$ and of its iterated derivatives everywhere in some neighbourhood of 0. None of them provides an exact identification method. This remark justifies our request of an algebraic identification.

3. Iterated derivatives of the output

For any series G, the time derivatives of the output are given by $y_G^{(i)}(t) = \langle G \,\|\, \mathscr{C}_a^{(i)}(t) \rangle$. Thus, we compute now the iterated time derivatives of the Chen's series (see [6]).

3.1. Derivatives of Chen series

Denoting by D_t the time derivation operator, we obtain recursively

$$\text{(R)} \quad \mathscr{C}_a^{(i)} = \mathscr{C}_a A_i, \quad \text{where} \quad \left\{ \begin{array}{ll} A_1 = \mathscr{L}_a & = \sum a_j z_j \\ A_{i+1} = \mathscr{L}_a A_i + D_t A_i & \end{array} \right\}.$$

We set $A_0 = 1$. Let us note $\mathscr{A}$ the formal sum of the A_i. We have the *combinatorial identity*

$$\sum_{i \in \mathbb{N}} A_i = \mathscr{A} = 1 + \mathscr{L}_a \mathscr{A} + D_t \mathscr{A} = \sum_{\rho} \alpha_\rho \mathscr{L}_a^{[\rho]},$$

where the 'derivative' $\mathscr{L}_a^{[\rho]}$ of $\mathscr{L}_a$ by a multiindex $\rho = (\rho_1, \ldots, \rho_p)$ is given as follows:

$$\mathscr{L}_a^{(\rho_k)} = \sum_i a_i^{(\rho_k)} z_i, \quad \mathscr{L}_a^{[\rho]} = \mathscr{L}_a^{(\rho_1)} \cdots \mathscr{L}_a^{(\rho_p)} = \sum_{i_1, \ldots, i_p} a_{i_1}^{(\rho_1)} \cdots a_{i_p}^{(\rho_p)} z_{i_1} \cdots z_{i_p}.$$

The *degree* of $\rho=(\rho_1,\dots,\rho_p)$ is p, the *weight* of ρ is the integer $\sum_{j=1}^{p}(1+\rho_j)=\rho_1+\cdots+\rho_k+k$. Then, we have $A_i=\sum_{\mathrm{wgt}(\rho)=i}\mathscr{L}_a^{[\rho]}$. An easy analysis of the combinatorial identity gives as result

$$\text{(E)}\quad \alpha_\rho=\prod_{i=1}^{p}\binom{\sum_{j=1}^{i}\rho_j+i-1}{\rho_i}=\binom{\rho_1}{\rho_1}\binom{\rho_1+\rho_2+1}{\rho_2}\cdots\binom{\rho_1+\cdots\rho_k+k-1}{\rho_k}.$$

3.2. Output derivatives and dynamical systems

With this definition of the coefficients $\alpha(\rho)$, we deduce

$$y^{(n)}(t)=\langle G\,\|\,\mathscr{C}_aA_n\rangle=\sum_{\mathrm{wgt}(\rho)=n}\alpha(\rho)\sum_{i_j}a_{i_1}^{(\rho_1)}(t)\cdots a_{i_k}^{(\rho_k)}(t)\langle z_{i_1}\cdots z_{i_k}\triangleleft G\,\|\,\mathscr{C}_a(t)\rangle,$$

$$y^{(n)}(0)=\langle G\,\|\,A_n\rangle=\sum_{\mathrm{wgt}(\rho)=n}\alpha(\rho)\sum_{i_j}a_{i_1}^{(\rho_1)}(0)\cdots a_{i_k}^{(\rho_k)}(0)\langle G\,\|\,z_{i_1}\cdots z_{i_k}\rangle.$$

In case of a *dynamical system* given by the state equations on an analytic manifold

$$(\Sigma)\quad\begin{cases}\dot q=g_0(q)+\sum_{i=1}^{m}a_i(t)g_i(q),\\ y(t)=h(q),\end{cases}$$

denoting by $g_i\circ h$ the Lie derivative of h with respect to the vector field g_i, the output of (Σ) is also the output $y_G(t)$ of the causal functional defined by its generating series G [3]

$$G=\sum_{k\geqslant 0}\sum_{i_j=0}^{m}g_{i_1}\cdots g_{i_k}\circ h_{|q_0}z_{i_1}\cdots z_{i_k}.$$

The formula giving the iterated derivative $y^{(n)}(t)$ may be interpreted by two means.

- A *local expression* in the state $q(t)$ (Lamnabhi-Lagarrigue and Crouch [8])

$$y^{(n)}(t)=\sum\sum\alpha(\rho)a_{i_1}^{(\rho_1)}(t)\cdots a_{i_k}^{(\rho_k)}(t)(g_{i_1}\cdots g_{i_k}\circ h)_{|q(t)}.$$

- A *global expression* in the state $q(0)$

$$y^{(n)}(t)=\sum\sum\alpha(\rho)a_{i_1}(0)^{(\rho_1)}\cdots a_{i_k}(0)^{(\rho_k)}y_{i_1i_2\cdots i_k}(t),$$

  where $y_{i_1i_2\cdots i_k}$ is the output of (Σ) initialized in $q(0)$, for the observation $g_{i_1}g_{i_2}\cdots g_{i_k}\circ h$.

(Other works on input/output differential equations can be found in [2,12,13].)

4. First step for computing the algebraic identification

In the expansion formula of $y^{(n)}(0)$, each product $a_{i_1}^{(\rho_1)}(0)\cdots a_{i_k}^{(\rho_k)}(0)$ can be viewed as the value in 0 of a monomial a^μ in the inputs a_i and their derivatives. So we get an expansion

$$y^{(n)}(0) = \sum_{\mathrm{wgt}(\mu)=n} a^{\mu} \langle G \| l_{\mu} \rangle.$$

Our aim is here to *effectively compute for each monomial* a^{μ}, *the contribution*: $\langle G \| l_{\mu} \rangle$.

4.1. Single input without drift

In this case, the identification problem has a completely straightforward solution as follows. We set $a_0(t) = c_0 \neq 0$ (constant). Then $A_1 = c_0 z_0$, $D_t A_i = 0$, $A_i = c_0^i z_0^i$, and finally,

$$y^{(n)}(0) = c_0^n \langle G | z_0^n \rangle.$$

Such a direct proof cannot be used for more than one input, or when there is a drift part.

4.2. Single input with drift

This case is concerned with dynamic systems having an autonomous part described by a 'non-controlled vector field' $g_o \neq 0$. Equivalently, we set $a_0(t) \equiv 1$. We take $a_1(t) = a(t) = \sum_{i=0}^{k} \frac{c_i}{i!} t^i$. (*In other words* $a^{(j)}(0) = c_j$, $\forall j \leqslant k$). The differential monomials take the form

$$a^{\nu} = (a^{(i_1)})^{e_1} (a^{(i_2)})^{e_2} \cdots (a^{(i_q)})^{e_q}.$$

The *weight* of ν *related to* a_1 is the sum $\sum_{j=1}^{q} e_j(1 + i_j)$. And then we get an expansion

$$y_a^{(n)} = \sum_{\mathrm{sum}_j \{e_j(1+i_j)\} \leqslant n} (a^{(i_1)})^{e_1} (a^{(i_2)})^{e_2} \cdots (a^{(i_q)})^{e_q} \; G(n)_{i_1, i_2, \ldots, i_q}^{e_1, e_2, \ldots, e_q} \qquad (\mathrm{FD})_n$$

We have to identify the coefficients $G(n)_{i_1, i_2, \ldots, i_q}^{e_1, e_2, \ldots, e_q}$. Such an equation $(\mathrm{FD})_n$ is provided for any input — that is for any free choice of the input coefficients c_i — and for the resulting output $y_a^{(n)}$.

1. *Identification of the* $G(n)_{\emptyset}^{\emptyset}$

   For the trivial input $a(t) \equiv 0$, we obtain directly $y^{(n)}(0) = \langle G | z_o^n \rangle = G(n)_{\emptyset}^{\emptyset}$.

2. *Identification of the* $G(n)_{i_1}^{e_1}$

   Define inputs by $c_j = \delta_{i_1, j} b_k$ for some constant b_k. The resulting output is

$$\sum_{e_1(1+i_1) \leqslant n} b_k^{e_1} G(n)_{i_1}^{e_1}.$$

By taking an adequate number of pairwise distinct k_j, the square-linear system has a Vandermonde determinant. Thus, we can identify the $G(n)_{i_1}^{e_1}$.

3. *Recurrence step*
We search to identify the $G(n)_{i_1,i_2,\dots,i_q}^{e_1,e_2,\dots,e_q}$ for a fixed set $I=\{i_1,i_2,\dots,i_q\}$ of lower indices. Suppose the already identified monomials for lower index set are *strictly* included in I. We use *specialized inputs* satisfying $c_j=0$ except in case $j\in I$. Thus, in $(\mathrm{FD})_n$ vanishes any monomial with some lower index j not in I. By eliminating the 'smaller' index sets (induction step), the linear system takes the form

$$L_n(a)=\sum_{\mathrm{sum}\{e_j(1+i_j)\}\leqslant n} c_{i_1}^{e_1}\cdots c_{i_q}^{e_q}\; G(n)_{i_1,i_2,\dots,i_q}^{e_1,e_2,\dots,e_q},$$

with $L_n(a)$ computed from related output and smaller index set eliminations. We associate to the 'column index' $G(n)_{i_1,i_2,\dots,i_q}^{e_1,e_2,\dots,e_q}$ the specialized input $A_{i_1,i_2,\dots,i_q}^{e_1,e_2,\dots,e_q}$ defined by

$$c_{i_1}=b_{i_1,e_1},\quad c_{i_2}=b_{i_2,e_2},\dots,\quad c_{i_q}=b_{i_q,e_q}.$$

The square matrix of the obtained linear system has the following properties:
(a) The row of index $A_{i_1,i_2,\dots,i_q}^{e_1,e_2,\dots,e_q}$ admits as common divisor the product

$$b_{i_1,e_1}b_{i_2,e_2}\cdots b_{i_q,e_q}.$$

(b) After factorizing these common divisors, straightforward equalities of rows occur when two distinct pairs $c_{i,j}$ and $c_{i',j'}$ are taken equal. Hence, the remaining determinant can be divided by a product of multiples of the differences $b_{i,j}=b_{i',j'}$, for $i\leqslant i'$, and $j\leqslant j'$. This last product and the determinant have the same polynomial degree. By checking the diagonal term of the determinant, we deduce that these two polynomials are equal.

By taking pairwise different values for the $c_{i,j}$, that achieves the proof.

Example. In the case $n=15$, let us study the $G(15)_{1,2,3}^{i,j,k}$. We obtain in the restricted expansion the following 9 terms, given below with their weights:

$$\mathrm{wgt}(G_{1,2,3}^{1,1,1})=9,\quad \mathrm{wgt}(G_{1,2,3}^{2,1,1})=11,\quad \mathrm{wgt}(G_{1,2,3}^{3,1,1})=13,$$

$$\mathrm{wgt}(G_{1,2,3}^{4,1,1})=15,\quad \mathrm{wgt}(G_{1,2,3}^{1,2,1})=12,\quad \mathrm{wgt}(G_{1,2,3}^{2,2,1})=14,$$

$$\mathrm{wgt}(G_{1,2,3}^{1,3,1})=15,\quad \mathrm{wgt}(G_{1,2,3}^{1,1,2})=13,\quad \mathrm{wgt}(G_{1,2,3}^{2,1,1})=15.$$

After the first-factorization process (a), the matrix of the *simplified linear system* takes the form

$G^{1,1,1}_{1,2,3}$	$G^{2,1,1}_{1,2,3}$	$G^{3,1,1}_{1,2,3}$	$G^{4,1,1}_{1,2,3}$	$G^{1,2,1}_{1,2,3}$	$G^{2,2,1}_{1,2,3}$	$G^{1,3,1}_{1,2,3}$	$G^{1,1,2}_{1,2,3}$	$G^{2,1,2}_{1,2,3}$
1	$b_{1,1}$	$b^2_{1,1}$	$b^3_{1,1}$	$b_{2,1}$	$b_{1,1}b_{2,1}$	$b^2_{2,1}$	$b_{3,1}$	$b_{1,1}b_{3,1}$
1	$b_{1,2}$	$b^2_{1,2}$	$b^3_{1,2}$	$b_{2,1}$	$b_{1,2}b_{2,1}$	$b^2_{2,1}$	$b_{3,1}$	$b_{1,2}b_{3,1}$
1	$b_{1,3}$	$b^2_{1,3}$	$b^3_{1,3}$	$b_{2,1}$	$b_{1,3}b_{2,1}$	$b^2_{2,1}$	$b_{3,1}$	$b_{1,3}b_{3,1}$
1	$b_{1,4}$	$b^2_{1,4}$	$b^3_{1,4}$	$b_{2,1}$	$b_{1,4}b_{2,1}$	$b^2_{2,1}$	$b_{3,1}$	$b_{1,4}b_{3,1}$
1	$b_{1,1}$	$b^2_{1,1}$	$b^3_{1,1}$	$b_{2,2}$	$b_{1,1}b_{2,2}$	$b^2_{2,2}$	$b_{3,1}$	$b_{1,1}b_{3,1}$
1	$b_{1,2}$	$b^2_{1,2}$	$b^3_{1,2}$	$b_{2,2}$	$b_{1,2}b_{2,2}$	$b^2_{2,2}$	$b_{3,1}$	$b_{1,2}b_{3,1}$
1	$b_{1,1}$	$b^2_{1,1}$	$b^3_{1,1}$	$b_{2,3}$	$b_{1,1}b_{2,3}$	$b^2_{2,3}$	$b_{3,1}$	$b_{1,1}b_{3,1}$
1	$b_{1,1}$	$b^2_{1,1}$	$b^3_{1,1}$	$b_{2,1}$	$b_{1,1}b_{2,1}$	$b^2_{2,1}$	$b_{3,2}$	$b_{1,1}b_{3,2}$
1	$b_{1,2}$	$b^2_{1,2}$	$b^3_{1,2}$	$b_{2,1}$	$b_{1,2}b_{2,1}$	$b^2_{2,1}$	$b_{3,2}$	$b_{1,2}b_{3,2}$

Its determinant is a polynomial of degree 14. By row comparisons (step (b)), we factorize

$$(b_{1,1}-b_{1,2})^3(b_{1,1}-b_{1,3})(b_{1,2}-b_{1,3})(b_{1,1}-b_{1,4})(b_{1,2}-b_{1,4})(b_{1,3}-b_{1,4})$$

$$(b_{2,1}-b_{2,2})^2(b_{2,1}-b_{2,3})(b_{2,2}-b_{2,3})(b_{3,1}-b_{3,2})^2$$

that is also a polynomial of degree 14. We check directly that the quotient is equal to 1.

4.3. Several inputs with drift

We consider the case of 2 inputs $a_1(t)$ and $a_2(t)$. We choose them as polynomials

$$a_1(t)=\sum_{i=0}^{l_1}\frac{c_i}{i!}t^i, \qquad a_2(t)=\sum_{i=0}^{l_2}\frac{d_i}{i!}t^i, \quad (\text{and } a_0(t)\equiv 1).$$

We obtain the following expansion:

$$y^{(n)}=\sum c_{i_1}^{e_1}\cdots c_{i_p}^{e_p}d_{j_1}^{f_1}\cdots d_{j_q}^{f_q}\ G(n)^{e_1,\dots,e_p f_1,\dots,f_q}_{i_1,\dots,i_p j_1,\dots,j_q}, \tag{FD$_n$}$$

where the sum is taken for $\mathrm{sum}\{e_k(1+i_k)\}+\mathrm{sum}\{f_k(1+j_k)\}\leqslant n$

The identification of the multiderivative coefficients appearing in the previous equation is solved by the same method as for a single-input system

1. Select some set $\{i_1,i_2,\dots,i_p;j_1,\dots,j_q\}$ of derivative orders, and suppose the identification already done for strict subsets of these derivatives. We use some *specialized inputs* satisfying $c_j=0$ (resp. $d_k=0$) except in case $j\in\{i_1,\dots,i_p\}$ *and* $k\in\{j_1,\dots,j_q\}$. We eliminate the 'shorter monomials' (by preceding identification steps).

To the 'column index' $G(n)_{i_1,\dots,i_p\,;\,j_1,\dots,j_q}^{e_1,\dots,e_q\,;\,f_1,\dots,f_q}$ we associate the specialized input defined by

$$c_{i_1}=b_{i_1,e_1},\dots,\quad c_{i_q}=b_{i_q,e_q},\quad \text{and}\quad d_{j_1}=v_{j_1,f_1},\dots,\quad d_{j_q}=v_{j_q,f_q}.$$

2. We can factorize in each row the corresponding product $b_{i_1,e_1}\dots b_{i_p,e_p}v_{j_1,f_1}\cdots v_{j_q,f_q}$. And thereafter, by purchasing in the simplified determinant the row equalities appearing each time that one equalizes two distinct indexed coefficients, we can completely compute the determinant, which vanishes only if one of the pairwise distinct coefficients b_{i_k,e_k} or v_{j_k,f_k} is also vanishing.

5. Conclusions

A first step has been taken towards algebraic identification. It remains, by using together the combinatorial definition of the multiderivatives $a^{(\rho)}$ and of the differential monomials a^{μ}, to separate the contribution $\langle G|w\rangle$ of each noncommutative word in G.

References

[1] K.T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831–879.

[2] P. Crouch, F. Lamnabhi-Lagarrigue, State realizations of nonlinear systems defined by input–output differential equations, Lecture Notes in Control and Inform. Sci. 11 (1988) 138–149.

[3] M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, Bull. Soc. Math. France 109 (1981) 3–40.

[4] M. Fliess, On the concept of derivatives and Taylor expansions for nonlinear input/output systems, Proc. 22nd IEEE Conf. Decision and Control, San Antonio, Texas, 1983, pp. 643–646.

[5] M. Fliess, M. Lamnabhi, F. Lamnabhi-Lagarrigue, An algebraic approach to nonlinear functional expansions, IEEE Trans. Circuits Systems CAS-30 (8) (1983) 554–570.

[6] C. Hespel, Iterated derivatives of a non linear dynamic system and Faà di Bruno formula, Math. Comput. Simulation, to appear.

[7] G. Jacob, Algebraic methods and computer algebra for nonlinear systems' study, in: IMACS Symposium MCTS, Modelling and Control of Technochological Systems, vol. 2, Lille, 1991, pp. 599–608.

[8] F. Lamnabhi-Lagarrigue, P.E. Crouch, A formula for iterated derivatives along trajectories of nonlinear systems, Systems Control Lett. 11 (1988) 1–7.

[9] C. Reutenauer, private communication.

[10] H.J. Sussmann, A product expansion for the Chen series, in: C. Byrnes, A. Lindquist (Eds.), Theory and Applications of Nonlinear Control Systems, Elsevier, Amsterdam, 1986, pp. 323–335.

[11] Y. Wang, E.D. Sontag, On two definitions of observation spaces, Systems Control Lett. 13 (1989) 279–289.

[12] Y. Wang, E.D. Sontag, Generating series and non linear systems: analytic aspects, local realizability and i/o representations, Forum Math. 4 (1992) 299–322.

[13] Y. Wang, E.D. Sontag, Algebraic differential equations and rational control systems, SIAM. J. Control Optim. 30 (5) (1992) 1126–1149.